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Determining the optimal process means under mixture normal distributions

Yen-Chang Chang · Cheng-Wen Chang · Wen-Liang Hung

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Abstract In this paper, we consider that the working environment has certain states, and in every state, the parameters of quality characteristics are different. Thus, if we set the characteristics parameters in a specified state, these parameters will change to another state. To describe this situation, we use a mixture of normal distributions, which comprise a flexible and powerful statistical-based modeling tool in practice. Under the step loss function and the piecewise linear loss function, we select the optimal means for the proposed manufacturing process.

Keywords Mixture normal distribution · Step loss function · Piecewise loss function

1 Introduction

Taguchi defines quality in terms of the loss imparted to society from the time that a product is shipped by the manufacturer. The basis of this definition is that the smaller the loss caused to society by a product, the better the product's quality. Viewing quality from a societal perspective is a profound perspective because it includes customers, manufacturers, and the broader community in the definition of quality. According to this perspective on quality, a quality improvement saves society more resources than it costs, and it benefits all parties: customers, manufactures, and the community. Hence,

Y.-C. Chang
Department of Applied Mathematics, National Hsinchu University of Education, Hsinchu City 300, Taiwan, ROC

C.-W. Chang
Department of Information Management, Nanya Institute of Technology, Chungli City 320, Taiwan, ROC

W.-L. Hung (✉)
Graduate Institute of Computer Science, National Hsinchu University of Education, Hsinchu City 300, Taiwan, ROC
e-mail: wlhung@mail.nhcue.edu.tw

investment in quality improvement is worthwhile so long as it reduces the loss to society from the time a product is shipped. The total loss to society to produce a product with given parameter values (nominal settings) consists of two component parts: (1) the production cost to the manufacturer of producing a product with given parameters, and (2) the inferior quality cost to the customer and community of producing a product with given parameters.

Any variation in a product's performance characteristics for its nominal value, at any randomly selected position in the product's life cycle, causes a loss to society. Again, let $L(y)$ equal the total cost to society as a result of a product's having a value of y for a specified performance characteristics, given that the nominal value for the performance characteristics is m. The simple type of loss function is the squared error, which is also referred to as the quadratic loss. That is $L(y) = (y-m)^2$. If the loss differs for values of y that are equidistant from m, for example, a value that exceeds m might be more detrimental than a value that is below m, then an asymmetric loss functions would be appropriate. In this paper we consider two asymmetric loss functions, one is the step loss function (e.g., Wen and Mergen 1999; Chen and Chou 2005) and the other is the piecewise linear loss function (e.g., Carlsson 1984; Golhar and Pollock 1998; Misiorek and Barnett 2000; Lee et al. 2001; Chen and Chou 2005).

In this paper, we consider that the working environment has certain states. In every state, the parameters of quality characteristics are different. Thus, if we set the characteristic parameters in one specified state, these parameters will change to another state. As an example, the parameters of quality characteristics may be different between the morning class and night class. Moreover, we assume the distribution of the quality characteristics are a mixture of normal distributions. That is, in each specified state the characteristics have specified normal distributions. For convenience, we assume there is only one characteristic, and we find that the optimal process mean under a mixture normal quality characteristic.

The remainder of this paper is organized as follows. In Sect. 2, the proposed mixture normal quality characteristic is introduced. In Sect. 3, we find the optimal parameters under the proposed model. Numerical examples are provided for illustration in Sect. 4. Conclusions are presented in the last section.

2 Mixture normal distributions

Finite mixture of distributions as an extremely flexible and powerful statistically-based modeling tool have received increasing attention from both practical and theoretical points of view. The literature surrounding them goes back to the famous biometrician Karl Pearson (1894), who considered the case of the mixture of two normal distributions and estimated the five parameters. McLachlan and Peel (2001) provided an up-to-date account of the theory and applications of modeling via finite mixture distributions. In particular, normal mixture models have been used extensively as models in a wide variety of practical situations where data can be viewed as arising from two or more populations mixed in varying proportions. Hence we shall consider the fitting of normal mixture models to the quality characteristics.

Assume that the process has p states and the probability of this process is in the ith state is α_i, for $i = 1, 2, \ldots, p$. Clearly we have $\sum_{i=1}^{p} \alpha_i = 1$ and $0 < \alpha_i < 1$ for all $i = 1, 2, \ldots, p$. The quality characteristic is normally distributed. If we set the process mean as μ, it will be shifted when the process in a difference state. The variance will also be changed in difference state. That is, in the ith state, the quality characteristic, X, is normally distributed with mean $\mu + \mu_i$ and variance σ_i^2, for $i = 1, 2, \ldots, p$. Note that the parameters are known as $-\infty < \mu_i < \infty$, $\sigma_i^2 > 0$, for $i = 1, 2, \ldots, p$. Then the probability density function of X is

$$f(x) = \sum_{i=1}^{p} \alpha_i f_i(x) = \sum_{i=1}^{p} \frac{\alpha_i}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{(x-\mu_i-\mu)^2}{2\sigma_i^2}\right],$$

where $f_i(x)$ is the probability density function of a normal distribution with mean $\mu + \mu_i$ and variance σ_i^2, $i = 1, 2, \ldots, p$.

If the shifts of the process mean, μ_i's, are close to zero and the shifts of process variance, σ_i^2's, are indistinct, then the model will close to the classical Taguchi (1986) quality model. To determine the optimal process mean, say $\mu = \mu^*$, by minimizing the expected cost of being out-of-specification, the loss function should be considered. For most decision analyses (cf. Cho and Leonard 1997; Wen and Mergen 1999; Chen and Chou 2005), the uses of step loss function and piecewise linear loss function make the calculations relatively straightforward and simple. Therefore, we use these two loss functions to measure the cost of being out-of-specification.

3 The optimal process mean

The tasks of finding the optimal process mean for mixture normal characteristic under the step loss function and the piecewise linear loss function are discussed as follows.

3.1 The step loss function

Let T_U be the upper specification limit, T_L be the lower specification limit, D_U be the monetary loss per item of exceeding T_U, and D_L be the monetary loss per item of staying below T_L. The step loss function is

$$L_1(\mu; T_\mathrm{U}, T_\mathrm{L}, D_\mathrm{U}, D_\mathrm{L}) = \begin{cases} D_\mathrm{U}, & \text{if } X > T_\mathrm{U}; \\ D_\mathrm{L}, & \text{if } X < T_\mathrm{L}. \end{cases}$$

Then the expected total loss per item is

$$\begin{aligned} C_1 = E[L_1] &= D_\mathrm{U} \int_{T_\mathrm{U}}^{\infty} f(x)\mathrm{d}x + D_\mathrm{L} \int_{-\infty}^{T_\mathrm{L}} f(x)\mathrm{d}x \\ &= D_\mathrm{U} \sum_{i=1}^{p} \left[\alpha_i \int_{T_\mathrm{U}}^{\infty} f_i(x)\mathrm{d}x\right] + D_\mathrm{L} \sum_{i=1}^{p} \left[\alpha_i \int_{-\infty}^{T_\mathrm{L}} f_i(x)\mathrm{d}x\right] \\ &= \sum_{i=1}^{p} \alpha_i \left\{ D_\mathrm{U}\left[1 - \Phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right] + D_\mathrm{L}\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)\right\}, \end{aligned} \tag{1}$$

where $\Phi(\cdot)$ is the cumulated distribution function of standard normal. Note that $D_\mathrm{U}[1 - \Phi(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i})] + D_\mathrm{L}\Phi(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i})$ is a convex function of μ for $i = 1, 2, \ldots, p$. Then Eq. 1 is a linear combination of convex functions, it implies C_1 is also a convex function of μ. Thus there is an optimal mean, say μ^*, such that C_1 is minimized.

Let $\phi(\cdot)$ be the probability density function of standard normal, i.e., $\phi \equiv \Phi'$. Using Eq. 1 we have

$$\begin{aligned}\frac{\partial C_1}{\partial \mu} &= \sum_{i=1}^{p} \alpha_i \left[D_{\mathrm{U}} \phi\left(\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i}\right) \frac{1}{\sigma_i} - D_{\mathrm{L}} \phi\left(\frac{T_{\mathrm{L}} - \mu_i - \mu}{\sigma_i}\right) \frac{1}{\sigma_i} \right] \\ &= D_{\mathrm{U}} \sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \phi\left(\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i}\right) - D_{\mathrm{L}} \sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \phi\left(\frac{T_{\mathrm{L}} - \mu_i - \mu}{\sigma_i}\right). \end{aligned} \tag{2}$$

If $\frac{\partial C_1}{\partial \mu} = 0$, we have

$$D_{\mathrm{U}} \sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \phi\left(\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i}\right) = D_{\mathrm{L}} \sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \phi\left(\frac{T_{\mathrm{L}} - \mu_i - \mu}{\sigma_i}\right).$$

Thus

$$\begin{aligned}\frac{D_{\mathrm{L}}}{D_{\mathrm{U}}} &= \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{U}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]} = \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{U}}-T_{\mathrm{L}}+T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]} \\ &= \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[\frac{-(T_{\mathrm{L}}-\mu_i-\mu)^2+(T_{\mathrm{U}}-T_{\mathrm{L}})^2+2(T_{\mathrm{U}}-T_{\mathrm{L}})(T_{\mathrm{L}}-\mu_i-\mu)}{2\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]} \\ &= \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right] \exp\left[-\frac{(T_{\mathrm{U}}-T_{\mathrm{L}})^2+2(T_{\mathrm{U}}-T_{\mathrm{L}})(T_{\mathrm{L}}-\mu_i-\mu)}{2\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]} \\ &= \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right] \exp\left[-\frac{(T_{\mathrm{U}}-T_{\mathrm{L}})=\left(\frac{T_{\mathrm{U}}+T_{\mathrm{L}}}{2}-\mu_i-\mu\right)}{2\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]} \\ &= \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right] \exp\left[-\frac{(T_{\mathrm{U}}-T_{\mathrm{L}})(\mu_i+\mu-\frac{T_{\mathrm{U}}+T_{\mathrm{L}}}{2})}{\sigma_i^2}\right]}{\sum_{i=1}^{p} \frac{\alpha_i}{\sigma_i} \exp\left[-\frac{(T_{\mathrm{L}}-\mu_i-\mu)^2}{2\sigma_i^2}\right]}. \end{aligned} \tag{3}$$

Then we can find the optimal mean μ^* by solving Eq. 3. The cases below are useful to find the optimal mean.

Case 1 If $\frac{D_{\mathrm{L}}}{D_{\mathrm{U}}} \geq 1$,

$$\begin{aligned}&\frac{\ln\left(\frac{D_{\mathrm{L}}}{D_{\mathrm{U}}}\right)}{T_{\mathrm{U}} - T_{\mathrm{L}}} \min_{1 \leq i \leq p} \sigma_i^2 + \frac{T_{\mathrm{U}} + T_{\mathrm{L}}}{2} - \max_{1 \leq i \leq p} \mu_i \leq \mu^* \\ &\quad \leq \frac{\ln(\frac{D_{\mathrm{L}}}{D_{\mathrm{U}}})}{T_{\mathrm{U}} - T_{\mathrm{L}}} \max_{1 \leq i \leq p} \sigma_i^2 + \frac{T_{\mathrm{U}} + T_{\mathrm{L}}}{2} - \min_{1 \leq i \leq p} \mu_i. \end{aligned} \tag{4}$$

Case 2 If $\frac{D_\mathrm{L}}{D_\mathrm{U}} < 1$,

$$\frac{\ln\left(\frac{D_\mathrm{L}}{D_\mathrm{U}}\right)}{T_\mathrm{U}-T_\mathrm{L}}\max_{1\le i\le p}\sigma_i^2+\frac{T_\mathrm{U}+T_\mathrm{L}}{2}-\max_{1\le i\le p}\mu_i\le\mu^*$$
$$\le\frac{\ln\left(\frac{D_\mathrm{L}}{D_\mathrm{U}}\right)}{T_\mathrm{U}-T_\mathrm{L}}\min_{1\le i\le p}\sigma_i^2+\frac{T_\mathrm{U}+T_\mathrm{L}}{2}-\min_{1\le i\le p}\mu_i. \tag{5}$$

3.2 The piecewise linear loss function

This subsection considers the piecewise loss function, which is

$$L_2(\mu;T_\mathrm{U},T_\mathrm{L},D_\mathrm{U},D_\mathrm{L})=\begin{cases}D_\mathrm{U}\ (X-T_\mathrm{U}), & \text{if } X>T_\mathrm{U};\\ D_\mathrm{L}\ (T_\mathrm{L}-X), & \text{if } X<T_\mathrm{L}.\end{cases},$$

where D_L is the quality loss coefficient when the quality characteristic is less than T_L and D_U is the quality loss coefficient when the quality characteristic exceeds T_U. The expected total loss per item is

$$C_2=E[L_2]=D_\mathrm{U}\int_{T_\mathrm{U}}^{\infty}(x-T_\mathrm{U})f(x)\mathrm{d}x+D_\mathrm{L}\int_{-\infty}^{T_\mathrm{L}}(T_\mathrm{L}-x)f(x)\mathrm{d}x$$
$$=D_\mathrm{U}\sum_{i=1}^{p}\left[\alpha_i\int_{T_\mathrm{U}}^{\infty}(x-T_\mathrm{U})f_i(x)\mathrm{d}x\right]+D_\mathrm{L}\sum_{i=1}^{p}\left[\alpha_i\int_{-\infty}^{T_\mathrm{L}}(T_\mathrm{L}-x)f_i(x)\mathrm{d}x\right]$$
$$=\sum_{i=1}^{p}\alpha_i\left\{D_\mathrm{U}\left[(\mu+\mu_i-T_\mathrm{U})\left[1-\Phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]+\sigma_i\phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]\right.$$
$$\left.+D_\mathrm{L}\left[(T_\mathrm{L}-\mu-\mu_i)\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)+\sigma_i\phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)\right]\right\} \tag{6}$$

The proof of the third equation of Eq. 6 is relegated to the Appendix. set

$$C_{2i}=D_\mathrm{U}\left[(\mu+\mu_i-T_\mathrm{U})\left[1-\Phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]+\sigma_i\phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]$$
$$+D_\mathrm{L}\left[(T_\mathrm{L}-\mu-\mu_i)\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)+\sigma_i\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)\right] \tag{7}$$

Then

$$\frac{\partial C_{2i}}{\partial\mu}=D_\mathrm{U}\left\{\left[1-\Phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]+(\mu+\mu_i-T_\mathrm{U})\phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\frac{1}{\sigma_i}\right.$$
$$\left.+\sigma_i\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(T_\mathrm{U}-\mu_i-\mu)^2}{2\sigma_i^2}\right)\frac{(T_\mathrm{U}-\mu_i-\mu)}{\sigma_i^2}\right]$$
$$+D_\mathrm{L}\left[-\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)-(T_\mathrm{L}-\mu_i-\mu)\phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right)\frac{1}{\sigma_i}\right.$$
$$\left.\left.+\sigma_i\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(T_\mathrm{L}-\mu_i-\mu)^2}{2\sigma_i^2}\right)\frac{(T_\mathrm{L}-\mu_i-\mu)}{\sigma_i^2}\right]\right\}$$
$$=D_\mathrm{U}\left[1-\Phi\left(\frac{T_\mathrm{U}-\mu_i-\mu}{\sigma_i}\right)\right]-D_\mathrm{L}\Phi\left(\frac{T_\mathrm{L}-\mu_i-\mu}{\sigma_i}\right) \tag{8}$$

Thus $\frac{\partial C_{2i}}{\partial \mu}$ is increasing in μ for $i = 1, 2, \ldots, p$. Moreover, we have $\lim_{\mu \to \infty} \frac{\partial C_{2i}}{\partial \mu} = D_{\mathrm{U}} > 0$ and $\lim_{\mu \to -\infty} \frac{\partial C_{2i}}{\partial \mu} = -D_{\mathrm{L}} < 0$ for $i = 1, 2, \ldots, p$. This implies $\frac{\partial C_2}{\partial \mu} = \sum_{i=1}^{p} \frac{\partial C_{2i}}{\partial \mu}$ is an increasing function of μ and $\lim_{\mu \to \infty} \frac{\partial C_2}{\partial \mu} = D_U < 0$ and $\lim_{\mu \to -\infty} \frac{\partial C_2}{\partial \mu} = -D_L < 0$. Thus there must exist a unique root of $\frac{\partial C_2}{\partial \mu}$, say μ^*, that can minimize the expected total loss per item C_2. Using Eq. 8, we can find μ^* by solving

$$\sum_{i=1}^{p} \alpha_i \left\{ D_{\mathrm{U}} \left[1 - \Phi \left(\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i} \right) \right] - D_{\mathrm{L}} \Phi \left[\frac{T_{\mathrm{L}} - \mu_i - \mu}{\sigma_i} \right] \right\} = 0. \tag{9}$$

4 Numerical examples

In this section, we consider two numerical examples to illustrate our results. Assume there are two states in the process. In state 1, $\mu = 0, \sigma_1 = 0.5$; in state 2, $\mu_2 = 0.2$, σ_2 =0.7. The probability of process is in the first state is $\alpha_1 = 0.8$. Then we have $\alpha_2 = 0.2$. The lower specification limit, T_{L} =2 and the upper specification limit, $T_{\mathrm{U}} = 4$. The monetary loss per item of falling below T_{L} is $D_{\mathrm{L}} = 1.5$. The monetary loss per item of exceeding T_{U} is $D_{\mathrm{U}} = 1$.

Example 1 (*The step loss*) As discussed in subsection 3.1, we have the bounds of μ^*. According to the assumptions above, using Eq. 4, we have $2.5507 \le \mu^* \le 3.0993$. By solving Eq. 3, we have $\mu^* = 2.7158$. The expected total cost is $C_1 = 0.1362$.

Example 2 (*The piecewise loss*) By solving Eq. 9, we have $\mu^* = 2.9898$. The expected total cost is $C_2 = 0.4100$.

5 Conclusions

In this study, we consider that the process parameters are shifted in different working environments. To describe these situations, we use a mixture of normal distributions which comprise a flexible and powerful statistical-based modeling tool in practice. Under the step loss function and the piecewise linear loss functions, we find the optimal means of the proposed manufacturing process. Moreover, two examples are used to illustrate our results. It is well known that the parameters of normal mixture of a two univariate normal homoscedastic components may be chosen so that US density is close in appearance to that of the log normal distribution (cf. McLachlan and Peel 2001). Therefore our results are more flexible than those of Chen and Chou (2005).

6 Appendix: The proof of (6)

It is enough to show that for all i,

$$\int_{T_{\mathrm{U}}}^{\infty} \frac{x - T_{\mathrm{U}}}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x - \mu_i - \mu)^2}{2\sigma_i^2} \right) \mathrm{d}x$$
$$= (\mu_i + \mu - T_{\mathrm{U}}) \left[1 - \Phi \left(\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i} \right) \right] + \sigma_i \phi \left[\frac{T_{\mathrm{U}} - \mu_i - \mu}{\sigma_i} \right] \tag{10}$$

and

$$\int_{-\infty}^{T_{\rm L}} \frac{T_{\rm L}-x}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x-\mu_i-\mu)^2}{2\sigma_i^2}\right) {\rm d}x$$
$$= (T_{\rm L}-\mu_i-\mu)\,\Phi\left(\frac{T_{\rm L}-\mu_i-\mu}{\sigma_i}\right) + \sigma_i\phi\left(\frac{T_{\rm L}-\mu_i-\mu}{\sigma_i}\right) \quad (11)$$

For convenience, we set $\mu_i \equiv 0$ and $\sigma_i \equiv \sigma$ as below.
Proof of (10)
Let Y = $(X-\mu)/\sigma$

$$\int_{T_{\rm U}}^{\infty} \frac{x-T_{\rm U}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) {\rm d}x = \int_{(T_{\rm U}-\mu)/\sigma}^{\infty} \frac{\sigma y+\mu-T_{\rm U}}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$
$$= (\mu-T_{\rm U})\left[1-\Phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right)\right] + \int_{(T_{\rm U}-\mu)/\sigma}^{\infty} \frac{\sigma y}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$
$$= (\mu-T_{\rm U})\left[1-\Phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right)\right] + \int_{|T_{\rm U}-\mu|/\sigma}^{\infty} \frac{\sigma y}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$

(since $y{\rm e}^{-y^2/2}$ is an odd function)

$$= (\mu-T_{\rm U})\left[1-\Phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right)\right] + \int_{(T_{\rm U}-\mu)^2/\sigma^2}^{\infty} \frac{\sigma}{2\sqrt{2\pi}} \exp\left(-\frac{u}{2}\right) {\rm d}u$$

$(u{=}y^2, {\rm d}u{=}2y{\rm d}y)$

$$= (\mu-T_{\rm U})\left[1-\Phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right)\right] + \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{(T_{\rm U}-\mu)^2}{2\sigma^2}\right)$$
$$= (\mu-T_{\rm U})\left[1-\Phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right)\right] + \sigma\phi\left(\frac{T_{\rm U}-\mu}{\sigma}\right).$$

Proof of (11)

$$\int_{-\infty}^{T_{\rm L}} \frac{T_{\rm L}-x}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) {\rm d}x = \int_{-\infty}^{(T_{\rm L}-\mu)/\sigma} \frac{T_{\rm L}-\sigma y-\mu}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$
$$= (T_{\rm L}-\mu)\,\Phi\left(\frac{T_{\rm L}-\mu}{\sigma}\right) - \int_{-\infty}^{(T_{\rm L}-\mu)/\sigma} \frac{\sigma y}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$
$$= (T_{\rm L}-\mu)\,\Phi\left(\frac{(T_{\rm L}-\mu)}{\sigma}\right) - \int_{-\infty}^{-|T_{\rm L}-\mu|/\sigma} \frac{\sigma y}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) {\rm d}y$$

(since $y{\rm e}^{-y^2/2}$ is an odd function)

$$= (T_{\rm L}-\mu)\,\Phi\left(\frac{T_{\rm L}-\mu^2}{\sigma^2}\right) - \int_{-\infty}^{(T_{\rm L}-\mu)/\sigma} \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{u}{2}\right) {\rm d}u$$

$(\text{u} = y^2,\ {\rm d}u = 2y{\rm d}y)$

$$= (T_{\rm L}-\mu)\,\Phi\left(\frac{T_{\rm L}-\mu}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{(T_{\rm L}-\mu)^2}{2\sigma^2}\right)$$
$$= (T_{\rm L}-\mu)\,\Phi\left(\frac{(T_{\rm L}-\mu)}{\sigma}\right) + \sigma\phi\left(\frac{T_{\rm L}-\mu}{\sigma}\right).$$

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